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SOME RESULTS ON H -AZUMAYA ALGEBRAS

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Let R be a commutative ring with identity, and let H be a finite Hopf algebra over R . In [3], F. Long defined the notion of an H -Azumaya algebra over R as a generalization of an Azumaya algebra over R . In this paper, we shall give some elementary properties of H -Azumaya algebras.

0. Preliminaries. Throughout this paper, R is a fixed commutative ring with identity, each \otimes is taken over R and each map is R -linear unless otherwise stated. Moreover H is a commutative cocommutative Hopf algebra over R , ε and Δ denote the counit and comultiplication maps of H respectively, and the action of Δ is denoted by $\Delta(h) = \sum_{(h)} h^{(1)} \otimes h^{(2)}$.

An R -algebra A is called an H -module algebra if A is an H -module such that the H -action map $\nu: H \otimes A \rightarrow A$ is an R -algebra map, that is, for $h \in H$, $a, b \in A$,

$$(0.1) \quad h(ab) = \sum_{(h)} (h^{(1)}a)(h^{(2)}b) \text{ and } h(1) = \varepsilon(h)1.$$

Similarly an R -algebra A is called an H -comodule algebra if A is an H -comodule via $\chi: A \rightarrow A \otimes H$ such that χ is an R -algebra map, that is, for $a, b \in A$,

$$(0.2) \quad \chi(ab) = \sum_{(a), (b)} a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)} \text{ and } \chi(1_A) = 1_A \otimes 1_H,$$

where $\chi(a) = \sum_{(a)} a^{(0)} \otimes a^{(1)}$. An R -algebra A is called an H -dimodule algebra if A is an H -module algebra and an H -comodule algebra such that the following diagram commutes

$$(0.3) \quad \begin{array}{ccc} H \otimes A & \xrightarrow{\nu} & A \\ 1 \otimes \chi \downarrow & & \downarrow \chi \\ H \otimes A \otimes H & \xrightarrow{\nu \otimes 1} & A \otimes H \end{array}$$

For an H -module algebra A and an H -comodule algebra B , the *smash product* $A \# B$ is equal to $A \otimes B$ as an R -module but with multiplication

$$(0.4) \quad (a_1 \# b_1)(a_2 \# b_2) = \sum_{(b_1)} a_1(b_1^{(1)}a_2) \# b_1^{(0)}b_2.$$

Moreover by [3, Th. 3.3], if A and B are H -dimodule algebras, then $A \# B$ is an H -dimodule algebra with the structure

$$(0.5) \quad h(a \# b) = \sum_{(h)} (h^{(1)}a) \# (h^{(2)}b)$$

and

$$(0.6) \quad \chi(a \# b) = \sum_{(a), (b)} a^{(0)} \# b^{(0)} \otimes a^{(1)} b^{(1)}.$$

For an H -dimodule algebra A , we define \bar{A} to be the R -module A with multiplication given by

$$(0.7) \quad \bar{a} \cdot \bar{b} = \overline{\sum_{(a)} (a^{(1)} b) a^{(0)}}$$

and with H -actions inherited from A . Then \bar{A} is really an H -dimodule algebra ([3, Th. 3.5]).

1. H -Azumaya algebra. In the following we shall always assume that A is an H -dimodule algebra.

Definition 1.1. An R -module M is called an H -dimodule left A -module if the following conditions are satisfied.

- (1) M is an H -dimodule and a left A -module.
- (2) $h(am) = \sum_{(h)} (h^{(1)}a)(h^{(2)}m)$ ($h \in H, a \in A, m \in M$).
- (3) $\chi(am) = \chi(a)\chi(m)$.

An H -dimodule right A -module is defined similarly.

Clearly the H -dimodule algebra A is an H -dimodule left A -module and an H -dimodule right A -module.

Now we define two maps

$$\begin{aligned} F: A \# \bar{A} &\longrightarrow \text{End}(A) \\ G: \bar{A} \# A &\longrightarrow \text{End}(A)^{\text{op}} \end{aligned}$$

by

$$\begin{aligned} F(a \# \bar{b})(c) &= \sum_{(b)} a(b^{(1)}c)b^{(0)} \\ G(\bar{a} \# b)(c) &= \sum_{(c)} (c^{(1)}a)c^{(0)}b. \end{aligned} \quad (a, b, c \in A)$$

By [3, Prop. 4.1], F and G are H -dimodule algebra maps. Then A is a left $A \# \bar{A}$ -module via

$$(1.1) \quad (a \# \bar{b})x = \sum_{(b)} a(b^{(1)}x)b^{(0)}$$

and a right $\bar{A} \# A$ -module via

$$(1.2) \quad x(\bar{a} \# b) = \sum_{(x)} (x^{(1)}a)x^{(0)}b.$$

Definition 1.2. ([3, Def. 4.2]). An H -dimodule algebra A is said to be H -Azumaya if it is a finitely generated projective faithful R -module and both F and G are isomorphisms.

Lemma 1.3. A is an H -dimodule left $A \# \bar{A}$ -module.

Proof. It is clear that A is an H -dimodule and left $A \# \bar{A}$ -module. Moreover since H is commutative and cocommutative, we have

$$\begin{aligned} h((a \# \bar{b})x) &= h(\sum_{(b)} a(b^{(1)}x)b^{(0)}) && \text{(by (1.1))} \\ &= \sum_{(h), (b)} (h^{(1)}a)(h^{(2)}(b^{(1)}x))(h^{(3)}b^{(0)}) && \text{(by (0.5))} \\ &= \sum_{(h), (b)} (h^{(1)}a)(b^{(1)}(h^{(3)}x))(h^{(2)}b^{(0)}) \\ &= \sum_{(h)} ((h^{(1)}a) \# (\overline{h^{(2)}b})) (h^{(3)}x) && \text{(by (0.3), (1.1))} \\ &= \sum_{(h)} (h^{(1)}(a \# \bar{b}))(h^{(2)}x) && \text{(by (0.1)).} \end{aligned}$$

This shows Def. 1.1 (2). Next, since H is commutative and cocommutative we have

$$\begin{aligned} \chi((a \# \bar{b})x) &= \sum_{(b)} \chi(a)\chi(b^{(1)}x)\chi(b^{(0)}) && \text{(by (1.1))} \\ &= \sum_{(a), (x), (b)} a^{(0)}(b^{(2)}x^{(0)})b^{(0)} \otimes a^{(1)}x^{(1)}b^{(1)} && \text{(by (0.2), (0.3))} \\ &= (\sum_{(a), (b)} (a^{(0)} \# \bar{b}^{(0)}) \otimes (a^{(1)}b^{(1)})) (\sum_{(x)} x^{(0)} \otimes x^{(1)}) && \text{(by (0.6))} \\ &= \chi(a \# \bar{b})\chi(x) && \text{(by (0.6)).} \end{aligned}$$

This shows Def. 1.1 (3). Hence A is an H -dimodule left $A \# \bar{A}$ -module.

Similarly we have the following

Lemma 1.3'. A is an H -dimodule right $\bar{A} \# A$ -module.

Definition 1.4. Let M be an H -dimodule left $A \# \bar{A}$ -module, and N an H -dimodule right $\bar{A} \# A$ -module. Given a subset S of A , we set

$$\begin{aligned} M^S &= \{m \in M \mid (s \# \bar{1})m = (1 \# \bar{s})m \text{ for all } s \in S\}, \\ {}^S N &= \{n \in N \mid n(\bar{1} \# s) = n(\bar{s} \# 1) \text{ for all } s \in S\}. \end{aligned}$$

Lemma 1.5. Let M be an H -dimodule left $A \# \bar{A}$ -module. Then the map $\phi: \text{Hom}_{A \# \bar{A}}(A, M) \longrightarrow M^A$ defined by $\phi(f) = f(1)$ is an R -module isomorphism.

Proof. For $f \in \text{Hom}_{A \# \bar{A}}(A, M)$, $a \in A$ and $m \in M^A$, we have

$$(a \# \bar{1})f(1) = f((a \# \bar{1})1) = f(a) = f((1 \# \bar{a})1) = (1 \# \bar{a})f(1).$$

Therefore, ϕ is well defined and monic. Now, for any $m \in M^A$, we

put $f_m(x) = (x \# \bar{1})m$ ($x \in A$). Then

$$\begin{aligned} (a \# \bar{b})f_m(x) &= (\sum_{(b)} a(b^{(1)}x) \# \overline{b^{(0)}})m && \text{(by (0.4), (0.7))} \\ &= \sum_{(b)} (a(b^{(1)}x) \# \bar{1})(1 \# \overline{b^{(0)}})m && \text{(by (0.4), (0.7))} \\ &= \sum_{(b)} (a(b^{(1)}x) \# \bar{1})(b^{(0)} \# \bar{1})m && \text{(by } m \in M^A) \\ &= (\sum_{(b)} (a(b^{(1)}x)b^{(0)}) \# \bar{1})m && \text{(by (0.4), (0.7))} \\ &= f_m((a \# \bar{b})x). \end{aligned}$$

Therefore f_m is in $\text{Hom}_{A \# \bar{A}}(A, M)$, that is, ϕ is an epimorphism.

Similarly, we have

Lemma 1.5'. *Let N be an H -dimodule right $\bar{A} \# A$ -module. Then the map $\phi' : \text{Hom}_{\bar{A} \# A}(A, N) \longrightarrow {}^A N$ defined by $\phi'(f) = f(1)$ is an R -module isomorphism.*

Corollary 1.6. (1) $\text{Hom}_{A \# \bar{A}}(A, A) \cong A^A$ as R -modules.

(2) $\text{Hom}_{\bar{A} \# A}(A, A) \cong {}^A A$ as R -modules.

Now we shall generalize the notion of a separable algebra in the next

Theorem 1.7. *If $\pi : A \# \bar{A} \longrightarrow A$ is defined by $\pi(a \# \bar{b}) = ab$, then the following are equivalent.*

(1) A is left $A \# \bar{A}$ -projective.

(2) There exists an element θ in $A \# \bar{A}$ such that $\pi(\theta) = 1$ and $(a \# \bar{1})\theta = (1 \# \bar{a})\theta$ for all $a \in A$.

Proof. First, we claim that π is a left $A \# \bar{A}$ -module epimorphism. In fact,

$$\begin{aligned} \pi((x \# \bar{y})(a \# \bar{b})) &= \pi(\sum_{(y)} x(y^{(2)}a) \# \overline{(y^{(1)}b)y^{(0)}}) && \text{(by (0.4), (0.7))} \\ &= \sum_{(y)} x(y^{(2)}a)(y^{(1)}b)y^{(0)} \\ &= \sum_{(y)} x(y^{(1)}(ab))y^{(0)} && \text{(by (0.1))} \\ &= (x \# \bar{y})\pi(a \# \bar{b}). \end{aligned}$$

(1) \Rightarrow (2). Since A is left $A \# \bar{A}$ -projective, there exists a left $A \# \bar{A}$ -module homomorphism $j : A \longrightarrow A \# \bar{A}$ such that $\pi j = 0$. If we put $\theta = j(1)$, then θ satisfies the condition (2).

(2) \Rightarrow (1). We put $\theta = \sum_i a_i \# \bar{b}_i$ and defined a map $j : A \longrightarrow A \# \bar{A}$ by $j(a) = (a \# \bar{1})\theta$. Then

$$(x \# \bar{y})j(a) = \sum_{i, (y)} x(y^{(2)}(aa_i)) \# \overline{(y^{(1)}b_i)y^{(0)}} \quad \text{(by (0.4), (0.7)).}$$

Noting that $(y \# \bar{1})\theta = (1 \# \bar{y})\theta$ for all $y \in A$, we have

$$(1.3) \quad \sum_i y a_i \# \bar{b}_i = \sum_{i, (v)} y^{(2)} a_i \# \overline{(y^{(1)} b_i) y^{(0)}} \quad (\text{by (0.4), (0.7)})$$

and

$$\begin{aligned} j((x \# \bar{y})a) &= \sum_{i, (v)} x(y^{(1)} a) y^{(0)} a_i \# \bar{b}_i && (\text{by (1.1)}) \\ &= \sum_{i, (v)} (x(y^{(1)} a) \# \bar{1})(y^{(0)} a_i \# \bar{b}_i) && (\text{by (0.4), (0.7)}) \\ &= \sum_{i, (v)} (x(y^{(3)} a) \# \bar{1})(y^{(2)} a_i \# \overline{(y^{(1)} b_i) y^{(0)}}) && (\text{by (1.3)}) \\ &= \sum_{i, (v)} x(y^{(3)} a)(y^{(2)} a_i) \# \overline{(y^{(1)} b_i) y^{(0)}} && (\text{by (0.4), (0.7)}) \\ &= \sum_{i, (v)} x(y^{(2)}(a a_i)) \# \overline{(y^{(1)} b_i) y^{(0)}} && (\text{by (0.1)}) \\ &= (x \# \bar{y})j(a) && (\text{by (0.4), (0.7)}). \end{aligned}$$

Therefore j is a left $A \# \bar{A}$ -module homomorphism, and $\pi j = 1$. Hence A is left $A \# \bar{A}$ -projective.

Similarly, we have the following

Theorem 1.7'. *If $\pi' : \bar{A} \# A \longrightarrow A$ is defined by $\pi'(\bar{a} \# b) = ab$, then the following are equivalent.*

- (1) A is right $\bar{A} \# A$ -projective.
- (2) There exists an element θ' in $\bar{A} \# A$ such that $\pi'(\theta') = 1$ and $\theta'(\bar{1} \# a) = \theta'(\bar{a} \# 1)$ for all $a \in A$.

Theorem 1.8. *The following conditions are equivalent.*

- (1) A is an R -progenerator and $F : A \# \bar{A} \longrightarrow \text{End}(A)$ is an isomorphism.
- (2) A is a left $A \# \bar{A}$ -progenerator and $A^A = R$.

Proof. (1) \Rightarrow (2). Since A is an R -progenerator and the map $F : A \# \bar{A} \longrightarrow \text{End}(A)$ is an isomorphism, A is a left $A \# \bar{A}$ -progenerator by [1, Cor. I. 3. 4]. Hence, by Cor. 1. 6 (1) we have

$$R \cong \text{Hom}_{\text{End}(A)}(A, A) \cong \text{Hom}_{A \# \bar{A}}(A, A) \cong A^A.$$

(2) \Rightarrow (1). This is clear by [1, Cor. I. 3. 4].

Similarly we have

Theorem 1.8'. *The following conditions are equivalent.*

- (1) A is an R -progenerator and $G : \bar{A} \# A \longrightarrow \text{End}(A)^{\text{op}}$ is an isomorphism.
- (2) A is a right $\bar{A} \# A$ -progenerator and ${}^A A = R$.

Remark 1.9. Let $Z(A)$ be the center of A , and let $Z(A)^H = \{z \in Z(A) \mid hz = \varepsilon(h)z \text{ for all } h \in H\}$. Then for any $z \in Z(A)^H$, $a \in A$,

we have

$$\begin{aligned}(1 \# \bar{a})z &= \sum_{(a)} (a^{(1)}z) a^{(0)} = \sum_{(a)} (\varepsilon(a^{(1)})z) a^{(0)} = z(\sum_{(a)} \varepsilon(a^{(1)}) a^{(0)}) \\ &= za = az = (a \# \bar{1})z.\end{aligned}$$

Therefore $R \subseteq Z(A)^H \subseteq A^A$. On the other hand, since

$$G_{\bar{1},1}(a) = \sum_{(a)} (a^{(1)}z) (a^{(0)}1) = za = az = G_{\bar{1},z}(a),$$

we have $\bar{z} \# 1 = \bar{1} \# z$ provided G is an isomorphism. Especially if G is an isomorphism and A is an R -progenerator, then z is in R . Hence if A satisfies the condition in Th. 1.8 or Th. 1.8', we have $R = Z(A)^H$.

As a combination of Th. 1.8, Th. 1.8' and Remark 1.9, we readily obtain the following

Theorem 1.10. *The following conditions are equivalent.*

- (1) A is H -Azumaya.
- (2) A is left $A \# \bar{A}$ -progenerator, right $\bar{A} \# A$ -progenerator and $A^A = R = Z(A)^H = {}^A A$.

2. Examples. In this section we shall give two examples of H -Azumaya algebras for which the Morita equivalence is also valid.

Let $R\text{-MOD}$ be the category of H -dimodules and H -module homomorphisms and let $A \# \bar{A}\text{-MOD}$ (resp. $\text{MOD-}\bar{A} \# A$) be the category of H -module left $A \# \bar{A}$ -(resp. right $\bar{A} \# A$ -) modules and H -module left $A \# \bar{A}$ -(resp. right $\bar{A} \# A$ -) module homomorphisms.

2.1. Let G be a group of order 2, and $H = RG$, the group algebra of G over R . If A is an H -module algebra, then for any $a \in A$, we have

$$\chi(a) = a_0 \otimes e + a_1 \otimes \sigma$$

where χ is the comodule structure map of A and $G = \{e, \sigma\}$ ($\sigma^2 = e$). Therefore for any $a, b \in A$, we have

- (2.1) $a = a_0 + a_1$ (unique) ($a_0, a_1 \in A$),
- (2.2) $(ab)_0 = a_0 b_0 + a_1 b_1, \quad (ab)_1 = a_0 b_1 + a_1 b_0,$
- (2.3) $(\sigma a)_0 = \sigma(a_0), \quad (\sigma a)_1 = \sigma(a_1).$

Throughout this subsection, we shall assume that $H = RG$, A is an H -module algebra and that M (resp. N) is an H -module left $A \# \bar{A}$ -(resp. right $\bar{A} \# A$ -) module.

Now, for $\text{Hom}_{A \# \bar{A}}(A, M)$ and $\text{Hom}_{\bar{A} \# A}(A, N)$ we define

$$\begin{aligned}
 (2.4) \quad & \begin{cases} (\sigma f)(a) = \sigma f(\sigma a) & (f \in \text{Hom}_{A \# \bar{A}}(A, M), a \in A), \\ \chi(f) = f_0 \otimes e + f_1 \otimes \sigma & \text{where } f_i(a) = (a \# \bar{1})f(1_i) \ (i = 1, 2), \end{cases} \\
 (2.5) \quad & \begin{cases} (\sigma f)(a) = \sigma(f(\sigma a)) & (f \in \text{Hom}_{\bar{A} \# A}(A, N), a \in A), \\ \chi(f) = f_0 \otimes e + f_1 \otimes \sigma & \text{where } f_i(a) = f(1_i)(1 \# \bar{a}) \ (i = 1, 2), \end{cases}
 \end{aligned}$$

respectively.

Proposition 2.1. $\text{Hom}_{A \# \bar{A}}(A, M)$ (resp. $\text{Hom}_{\bar{A} \# A}(A, N)$) is an H -dimodule concerning the structure (2.4) (resp. (2.5)) and $\text{Hom}_{A \# \bar{A}}(A, M) \cong M^A$ (resp. $\text{Hom}_{\bar{A} \# A}(A, N) \cong {}^A N$) as H -dimodules, where the H -dimodule structure of M^A (resp. ${}^A N$) inherits from M (resp. N).

Proof. First, we prove that M^A is an H -subdimodule of M . Let $m \in M^A$, $a \in A$. Then $\sigma((\sigma a \# \bar{1})m) = \sigma((1 \# \bar{\sigma a})m)$, and so σm is in M^A . Since $(a_i \# \bar{1})m = (1 \# \bar{a}_i)m$ ($i = 1, 2$), we have $\chi((a_i \# \bar{1})m) = \chi((1 \# \bar{a}_i)m)$. By (2.1) we obtain $m_0, m_1 \in M^A$. Thus M^A is an H -subdimodule of M . Similarly ${}^A N$ is an H -subdimodule of N .

Since M is an H -dimodule left $A \# \bar{A}$ -module, $f(1) \in M^A$ and $f(1)_i \in M^A$ by Lemma 1.5. Now we can easily see that $\text{Hom}_{A \# \bar{A}}(A, M)$ is an H -dimodule. It remains therefore to show that the map $\phi: \text{Hom}_{A \# \bar{A}}(A, M) \rightarrow M^A$ defined by $\phi(f) = f(1)$ is an H -dimodule isomorphism. But this is easy by the definition of f_i and Lemma 1.5. Similarly $\text{Hom}_{\bar{A} \# A}(A, N) \cong {}^A N$ as H -dimodules.

Remark 2.2. The Morita theory for $Z/2Z$ -graded case (resp. G -graded case) is developed in [2] (resp. [5]). In [2] (resp. [5]), if we define the G -action on A by $\sigma a = a$ ($\sigma \in G$, $a \in A$) then each $Z/2Z$ -graded (resp. G -graded) Azumaya algebra is an H -Azumaya by [3, p. 588]. Therefore the following theorem is a generalization of the Morita theory for $Z/2Z$ -graded case. (Recently in [4], M. Orzech announced that a Morita theory for G -dimodules is developed by M. Beattie.)

Theorem 2.3. If A is H -Azumaya, then each of the following pairs of functors establishes an isomorphism of categories:

$$\begin{aligned}
 (1) \quad & \begin{aligned} \mathcal{F} : R\text{-MOD} &\longrightarrow A \# \bar{A}\text{-MOD} & \mathcal{F}(X) &= A \otimes X, \\ \mathcal{G} : A \# \bar{A}\text{-MOD} &\longrightarrow R\text{-MOD}, & \mathcal{G}(Y) &= Y^A. \end{aligned} \\
 (2) \quad & \begin{aligned} \mathcal{H} : R\text{-MOD} &\longrightarrow \text{MOD} \cdot \bar{A} \# A, & \mathcal{H}(X) &= X \otimes A, \\ \mathcal{L} : \text{MOD} \cdot \bar{A} \# A &\longrightarrow R\text{-MOD}, & \mathcal{L}(Y) &= {}^A Y. \end{aligned}
 \end{aligned}$$

Proof. We shall prove only (1), and leave (2) to the reader. Since

A is an R -progenerator and $A \# \bar{A} \cong \text{End}(A)$, by Morita theory [1, Prop. I. 3. 3] there hold

$$\text{Hom}_{A \# \bar{A}}(A, A \# \bar{A}) \cong \text{Hom}(A, R) \text{ and } R \cong \text{Hom}(A, R) \otimes_{A \# \bar{A}} A.$$

Thus we have

$$\begin{aligned} X &\cong \text{Hom}(A, R) \otimes_{A \# \bar{A}} (A \otimes X) \cong \text{Hom}_{A \# \bar{A}}(A, A \# \bar{A}) \otimes_{A \# \bar{A}} (A \otimes X) \\ &\cong \text{Hom}_{A \# \bar{A}}(A, A \# \bar{A} \otimes_{A \# \bar{A}} (A \otimes X)) \\ &\quad \text{(by [1, I. 2. 7])} \\ &\cong \text{Hom}_{A \# \bar{A}}(A, A \otimes X) \cong (A \otimes X)^A \\ &\quad \text{(by Prop. 2. 1),} \end{aligned}$$

and similarly

$$\begin{aligned} Y &\cong A \otimes \text{Hom}_{A \# \bar{A}}(A, A \# \bar{A}) \otimes_{A \# \bar{A}} Y \cong A \otimes \text{Hom}_{A \# \bar{A}}(A, A \# \bar{A} \otimes_{A \# \bar{A}} Y) \\ &\cong A \otimes \text{Hom}_{A \# \bar{A}}(A, Y) \cong A \otimes Y^A \\ &\quad \text{(by Prop. 2. 1).} \end{aligned}$$

2. 2. Let R be a commutative algebra over $GF(2)$. Let $H = R \oplus R\delta$ be a free R -module with a free basis $\{1, \delta\}$. Then H is a Hopf algebra with the following algebra and coalgebra structure

$$\delta^2 = 0, \quad \varepsilon(\delta) = 0, \quad \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta.$$

If A is an H -dimodule algebra, then for any $a, b \in A$, we have

$$(2.6) \quad \partial(ab) = (\partial a)b + a(\partial b),$$

$$(2.7) \quad \chi(a) = a \otimes 1 + a_1 \otimes \delta, \quad \chi(a_1) = a_1 \otimes 1 \quad (a_1 \in A),$$

$$(2.8) \quad (ab)_1 = ab_1 + a_1b,$$

$$(2.9) \quad \chi(\partial a) = \partial a \otimes 1 + \partial a_1 \otimes \delta, \quad (\partial a)_1 = \partial a_1.$$

Throughout this subsection, we shall assume that $H = R \oplus R\delta$, A is an H -dimodule algebra and that M (resp. N) is an H -dimodule left $\bar{A} \# A$ - (resp. right $\bar{A} \# A$ -) module.

Now, for $\text{Hom}_{A \# \bar{A}}(A, M)$ and $\text{Hom}_{\bar{A} \# A}(A, N)$ we define

$$(2.10) \quad \begin{cases} \partial(f)(a) = \partial(f(a)) + f(\partial a) \\ \chi(f) = f \otimes 1 + f_1 \otimes \delta, \end{cases} \quad \begin{aligned} &(f \in \text{Hom}_{A \# \bar{A}}(A, M), \quad a \in A), \\ &\text{where } f_1(a) = (a \# \bar{1})f(1)_1, \end{aligned}$$

$$(2.11) \quad \begin{cases} (\partial f)(a) = \partial(f(a)) + f(\partial a) \\ \chi(f) = f \otimes 1 + f_1 \otimes \delta, \end{cases} \quad \begin{aligned} &(f \in \text{Hom}_{\bar{A} \# A}(A, N), \quad a \in A), \\ &\text{where } f_1(a) = f(1)_1(\bar{1} \# a), \end{aligned}$$

respectively.

Proposition 2.4. $\text{Hom}_{A \sharp \bar{A}}(A, M)$ (resp. $\text{Hom}_{\bar{A} \sharp A}(A, N)$) is an H -dimodule concerning the structure (2.10) (resp. (2.11)) and $\text{Hom}_{A \sharp \bar{A}}(A, M) \cong M^A$ (resp. $\text{Hom}_{\bar{A} \sharp A}(A, N) \cong {}^A N$) as H -dimodules, where the H -dimodule structure of M^A (resp. ${}^A N$) inherits from M (resp. N).

Proof. First, we prove that M^A is an H -subdimodule of M . Let $m \in M^A$, $a \in A$. Then by $\delta((a \# \bar{1})m) = \delta((1 \# \bar{a})m)$ and $(\delta a \# \bar{1})m = (1 \# \delta a)m$, we have $m \in M^A$. Moreover, by $\chi((a \# \bar{1})m) = \chi((1 \# \bar{a})m)$ and $(a_1 \# \bar{1})m = (1 \# \bar{a}_1)m$, we have $m_1 \in M^A$. Therefore M^A is an H -subdimodule of M . Similarly ${}^A N$ is an H -subdimodule of N .

Next, we show that $\text{Hom}_{A \sharp \bar{A}}(A, M)$ is an H -module and an H -comodule. Let $a, b, x \in A$, $f \in \text{Hom}_{A \sharp \bar{A}}(A, M)$. Then

$$\begin{aligned} (a \# \bar{b})((\delta f)(x)) &= (a \# \bar{b})(\delta((x \# \bar{1})f(1)) + (\delta x \# \bar{1})f(1)) \\ &= (a \# \bar{b})(\delta(x \# \bar{1})f(1) + (x \# \bar{1})\delta(f(1)) + (\delta x \# \bar{1})f(1)) \\ &\quad \text{(by Def. 1.1(2), (0.5), (2.6))} \\ &= (a \# \bar{b})(x \# \bar{1})\delta(f(1)) \end{aligned}$$

and

$$\begin{aligned} (\delta f)((a \# \bar{b})x) &= \delta(f((a \# \bar{b})x)) + f(\delta((a \# \bar{b})x)) \\ &= \delta((a \# \bar{b})(x \# \bar{1})f(1)) + f((\delta(a \# \bar{b})x + (a \# \bar{b})\delta x)) \\ &\quad \text{(by Def. 1.1(2))} \\ &= (a \# \bar{b})(x \# \bar{1})\delta(f(1)) \\ &\quad \text{(by (0.5), (2.6), } f \in \text{Hom}_{A \sharp \bar{A}}(A, M)). \end{aligned}$$

Therefore $f \in \text{Hom}_{A \sharp \bar{A}}(A, M)$. Moreover,

$$\begin{aligned} (a \# \bar{b})(f_1(x)) &= \sum_{(b)} (a(b^{(1)}x) \# \bar{b}^{(0)})(f(1))_1 && \text{(by (0.4), (0.6))} \\ &= \sum_{(b)} (a(b^{(1)}x) \# \bar{1})(1 \# \bar{b}^{(0)})(f(1))_1 && \text{(by (0.4), (0.6))} \\ &= \sum_{(b)} (a(b^{(1)}x) \# \bar{1})(b^{(0)} \# \bar{1})(f(1))_1 && \text{(by } f(1)_1 \in M^A) \\ &= \sum_{(b)} (a(b^{(1)}x)b^{(0)} \# \bar{1})(f(1))_1 && \text{(by (0.4), (0.6))} \\ &= f_1(\sum_{(b)} a(b^{(1)}x)b^{(0)}) = f_1((a \# \bar{b})x). \end{aligned}$$

Hence we have $f_1 \in \text{Hom}_{A \sharp \bar{A}}(A, M)$. Then it is easy to see that $\text{Hom}_{A \sharp \bar{A}}(A, M)$ is an H -module and an H -comodule. It remains therefore to show that $\text{Hom}_{A \sharp \bar{A}}(A, M)$ is an H -dimodule. In fact, we have

$$\begin{aligned} (\delta f_1)(x) &= \delta(f_1(x)) + f_1(\delta x) = \delta((x \# \bar{1})(f(1))_1) + (\delta x \# \bar{1})(f(1))_1 \\ &= (x \# \bar{1})\delta((f(1))_1) && \text{(by (0.5), Def. 1.1(2))} \\ &= (x \# \bar{1})((\delta f)(1))_1 = (\delta f)_1(x), \end{aligned}$$

and $\chi(\delta f) = f \otimes 1 + f_1 \otimes \delta$. Hence $\text{Hom}_{A \# \bar{A}}(A, M)$ is an H -dimodule.

Finally, we show that the map $\phi : \text{Hom}_{A \# \bar{A}}(A, M) \longrightarrow M^A$ defined by $\phi(f) = f(1)$ is an H -dimodule isomorphism. In fact,

$$\phi(\delta f) = (\delta f)(1) = \delta f(1) + f(\delta 1) = \delta f(1) = \delta \phi(f)$$

and

$$\begin{aligned} \chi(\phi(f)) &= \chi(f(1)) = f(1) \otimes 1 + (f(1))_1 \otimes \delta = f(1) \otimes 1 + f_1(1) \otimes \delta \\ &= (\phi \otimes 1)\chi(f). \end{aligned}$$

Hence ϕ is an H -dimodule isomorphism by Lemma 1.5. Similarly $\text{Hom}_{\bar{A} \# A}(A, N) \cong {}^A N$ as H -dimodules.

By Prop. 2.3, Morita theory [1. Cor. I. 3. 4] and the proof of Th. 2.2, we have the following

Theorem 2.5. *If A is H -Azumaya, then each of the following pairs of functors establishes an isomorphism of categories:*

$$\begin{aligned} (1) \quad \mathcal{F} : R\text{-MOD} &\longrightarrow A \# \bar{A}\text{-MOD}, & \mathcal{F}(X) &= A \otimes X, \\ \mathcal{G} : A \# \bar{A}\text{-MOD} &\longrightarrow R\text{-MOD}, & \mathcal{G}(Y) &= Y^A. \\ (2) \quad \mathcal{H} : R\text{-MOD} &\longrightarrow \text{MOD-}\bar{A} \# A, & \mathcal{H}(X) &= X \otimes A, \\ \mathcal{L} : \text{MOD-}\bar{A} \# A &\longrightarrow R\text{-MOD}, & \mathcal{L}(Y) &= {}^A Y. \end{aligned}$$

REFERENCES

- [1] F. DEMEYER and E. INGRAHAM: Separable Algebras over Commutative Ring, Lecture Notes in Math. **181**, Springer, Berlin, 1971.
- [2] L. N. CHILDS, G. GARFINKEL and M. ORZEC: The Brauer group of graded Azumaya algebras, Trans. Amer. Math. Soc. **175**(1973), 299—326.
- [3] F. W. LONG: The Brauer group of dimodule algebras, J. Alg. **31** (1974), 559—601.
- [4] M. ORZEC: On the Brauer group of algebras having a grading and an action, Canad. J. Math. **28** (1976), 533—552.

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